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ON THE PERTURBATIONS OF SMALL ECCENTRICITY SATELLITES

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SUMMARY

For small eccentricity satellites, so-called "first-order" orbit theories actually contain errors of the first order by neglecting J_2^2 , etc. terms having eccentricity as a divisor. These terms appear in the expressions representing the perturbations in eccentricity, mean anomaly, and argument of perigee. Using the von Zeipel method, terms of this sort through J_2^3 and $(J_i/J_2)^3$, J_i being any odd zonal harmonic coefficient, are presented here. Amplitudes of the terms for the satellites Alouette 1 and Tiros 8 are given. In addition, it is shown that these "small divisor" terms produce no in-track, along-track, or cross-track errors.

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INTRODUCTION

For most artificial earth satellites, orbit theories expressing in analytic form the perturbations in orbital elements caused by the non-sphericity of the earth do not suffer appreciably in accuracy if terms involving only the first power of J_2 are considered. However, for satellites whose eccentricities are approximately the same magnitude as J_2 ($\sim 10^{-3}$), such theories are, in effect, disregarding terms of the same order as the purported "accuracy" of the theories. The reason for this is the appearance of eccentricity as a divisor in some of the "higher-order" terms (i.e., J_2^2 , J_2^3 , J_3^2/J_2^2 , etc.), both short and long period, representing the perturbations in eccentricity, mean anomaly, and argument of perigee. As a result, many authors have proposed the adoption of different sets of variables to be used in solving the perturbation equations. However, if the eccentricity is not too small, these terms can be used with a reasonable amount of confidence.

In this paper will be presented those terms involving up to J_2^3 and $(J_i/J_2)^3$ (where J_i is an odd zonal harmonic), and having eccentricity as a divisor, in the perturbation expressions for eccentricity, mean anomaly, and argument of perigee. In addition, it will be drawn that these small divisors do not cause in-track, cross-track, or along-track errors. The von Zeipel method will be used, so basically this will be an abbreviated extension of the work done by Brouwer (Reference 1) and Kozai (Reference 2). Amplitudes of the short-period terms will be shown for the satellites Alouette 1 and Tiros 8. Also, the results of Reference 3 will be repeated, with a slight modification in the expression for the eccentricity. A list of symbols appears at the end of the report.

SHORT PERIOD PERTURBATIONS IN ℓ , g

The necessary parts of the determining functions S_1 and S_2 (see Reference 2) are

$$S_1 \cong \frac{J_2}{16a' \sqrt{a'}} (2 + 3e'^2) \left\{ -2(1 - 3\cos^2 i') (f - \ell + e' \sin f) + \sin^2 i' [3e' \sin(f + 2g) + 3 \sin 2(f + g) + e' \sin(3f + 2g)] \right\}$$

$$\begin{aligned} S_2 \cong & \frac{J_2^2}{256a'^3 \sqrt{a'}} (-48 \cos^2 i' (1 - 5\cos^2 i') (f - \ell) + 6e' (29 - 106 \cos^2 i' + 181 \cos^4 i') \sin f \\ & + 6(13 - 50 \cos^2 i' + 61 \cos^4 i') \sin 2f + 18e' (7 - 22 \cos^2 i' + 23 \cos^4 i') \sin 3f \\ & + 12e' \sin^2 i' (11 - 69 \cos^2 i') \sin(f + 2g) - 72e' \sin^2 i' (1 - 3 \cos^2 i') \sin(f - 2g) \\ & - 27e' \sin^4 i' \sin(f + 4g) - 24 \sin^2 i' (1 + 3 \cos^2 i') \sin 2(f + g) - 9 \sin^4 i' \sin(2f + 4g) \\ & - 4e' \sin^2 i' (47 - 121 \cos^2 i') \sin(3f + 2g) - 3e' \sin^2 i' (49 - 73 \cos^2 i') \sin(3f + 4g) \\ & - 84 \sin^2 i' (1 - 3 \cos^2 i') \sin(4f + 2g) - 12 \sin^2 i' (5 - 8 \cos^2 i') \sin 4(f + g) \\ & - 96e' \sin^2 i' (1 - 3 \cos^2 i') \sin(5f + 2g) + 3e' \sin^2 i' (21 - 13 \cos^2 i') \sin(5f + 4g) \\ & + 49 \sin^4 i' \sin(6f + 4g) + 63e' \sin^4 i' \sin(7f + 4g) + 36 \sin^2 i' (1 - 3 \cos^2 i') \sin 2g) \end{aligned}$$

S_2 contributes nothing to the "small divisor" perturbations in ℓ and g ; however, it will provide a contribution to the perturbation in e .

Since

$$\ell = \ell' - \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{L}'}$$

$$\mathbf{g} = \mathbf{g}' - \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{G}'}, \quad (2)$$

a transformation to primed variables by means of a Taylor Series expansion gives (inclusive of J_2^3 terms)

$$\begin{aligned} \ell = \ell' &= -\frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{L}'} - \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{L}' \partial \mathbf{f}'} - \left[-\frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{L}' \partial \mathbf{f}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{L}' \partial \mathbf{g}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{L}'} \right] \\ &= -\frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{L}' \partial \mathbf{g}'} \left[-\frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial \mathbf{f}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial \mathbf{g}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{G}'} \right] - \frac{1}{2} \frac{\partial^3 \mathbf{S}_1}{\partial \mathbf{L}' \partial \mathbf{f}'^2} \left(\frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} \right)^2 \\ &\quad - \frac{1}{2} \frac{\partial^3 \mathbf{S}_1}{\partial \mathbf{L}' \partial \mathbf{g}'^2} \left(\frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} \right)^2 - \frac{\partial^3 \mathbf{S}_1}{\partial \mathbf{L}' \partial \mathbf{f}' \partial \mathbf{g}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} + \frac{\partial^2 \mathbf{S}_2}{\partial \mathbf{L}' \partial \mathbf{f}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_2}{\partial \mathbf{L}' \partial \mathbf{g}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} \\ \mathbf{g} = \mathbf{g}' &= -\frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{G}'} - \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial \mathbf{f}'} - \left[-\frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial \mathbf{f}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial \mathbf{g}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{G}'} \right] \\ &= -\frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial \mathbf{g}'} \left[-\frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial \mathbf{f}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial \mathbf{g}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{G}'} \right] - \frac{1}{2} \frac{\partial^3 \mathbf{S}_1}{\partial \mathbf{G}' \partial \mathbf{f}'^2} \left(\frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} \right)^2 \\ &\quad - \frac{1}{2} \frac{\partial^3 \mathbf{S}_1}{\partial \mathbf{G}' \partial \mathbf{g}'^2} \left(\frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} \right)^2 - \frac{\partial^3 \mathbf{S}_1}{\partial \mathbf{G}' \partial \mathbf{f}' \partial \mathbf{g}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} + \frac{\partial^2 \mathbf{S}_2}{\partial \mathbf{G}' \partial \mathbf{f}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_2}{\partial \mathbf{G}' \partial \mathbf{g}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} \end{aligned} \quad (3)$$

where f and g are to be replaced by f' and g' in Equations (3). Using the relations

$$\frac{\partial f'}{\partial L'} = \frac{1}{e'} \frac{G'^2}{L'^3} \left(\frac{a'}{r'} + \frac{L'^2}{G'^2} \right) \sin f'$$

$$\frac{\partial f'}{\partial G'} = -\frac{1}{e'} \frac{G'}{L'^2} \left(\frac{a'}{r'} + \frac{L'^2}{G'^2} \right) \sin f'$$

$$\frac{d f'}{d \ell'} = \frac{G'}{L'} \frac{a'^2}{r'^2}$$

$$\frac{a'}{r'} = \frac{L'^2}{G'^2} (1 + e' \cos f')$$

$$\frac{\partial e'}{\partial L'} = -\frac{1}{e'} \frac{G'^2}{L'^3}$$

$$\frac{\partial e'}{\partial G'} = -\frac{1}{e'} \frac{G'}{L'^2} \quad (4)$$

and retaining only terms containing the largest powers of e' in the denominators, Equations (3) become

$$\dot{\ell} = \ell' + \Delta \ell,$$

where

$$\begin{aligned} \Delta \ell = & \frac{J_2}{8e'a'^2} [6(1 - 3\cos^2 i') \sin f' + 3\sin^2 i' \sin(f' + 2g') - 7\sin^2 i' \sin(3f' + 2g')] \\ & + \frac{J_2^2}{128e'^2 a'^4} [36\sin^2 i' (1 - 3\cos^2 i') \sin 2g' + 6(13 - 50\cos^2 i' + 61\cos^4 i') \sin 2f' \\ & - 9\sin^4 i' \sin(2f' + 4g') - 84\sin^2 i' (1 - 3\cos^2 i') \sin(4f' + 2g')] \\ & + 49\sin^4 i' \sin(6f' + 4g')] \\ & - \frac{J_2^3}{4096e'^3 a'^6} [12(1 - 3\cos^2 i') (97 - 266\cos^2 i' + 241\cos^4 i') \sin f' \\ & - 324(1 - 3\cos^2 i') (9 - 26\cos^2 i' + 25\cos^4 i') \sin 3f' + 3\sin^2 i' (263 - 1150\cos^2 i' + 1511\cos^4 i') \sin(f' + 2g')] \end{aligned}$$

$$\begin{aligned}
& +81 \sin^2 i' (19 - 86 \cos^2 i' + 115 \cos^4 i') \sin(f' - 2g') + 486 \sin^4 i' (1 - 3 \cos^2 i') \sin(f' + 4g') \\
& - \sin^2 i' (1081 - 4610 \cos^2 i' + 5977 \cos^4 i') \sin(3f' + 2g') - 306 \sin^4 i' (1 - 3 \cos^2 i') \sin(3f' + 4g') \\
& - 81 \sin^6 i' \sin(3f' + 6g') + 189 \sin^2 i' (19 - 86 \cos^2 i' + 115 \cos^4 i') \sin(5f' + 2g') \\
& + 546 \sin^4 i' (1 - 3 \cos^2 i') \sin(5f' + 4g') + 63 \sin^6 i' \sin(5f' + 6g') \\
& - 2646 \sin^4 i' (1 - 3 \cos^2 i') \sin(7f' + 4g') - 147 \sin^6 i' \sin(7f' + 6g') + 1029 \sin^6 i' \sin(9f' + 6g')
\end{aligned}$$

$$g = g' + \Delta g,$$

where

$$\Delta g = -\Delta t. \quad (5)$$

It will be noted that no divisors of e' appear in Δg ; it will also be noted that a long period ($\sin 2g'$) term has appeared in Δt and Δg .

SHORT PERIOD PERTURBATIONS IN e

Writing

$$\begin{aligned}
L &= L' + \Delta L \\
G &= G' + \Delta G,
\end{aligned} \quad (6)$$

a Taylor Series expansion for e gives (inclusive of J_2^3 terms)

$$\begin{aligned}
e &= e' + \Delta L \frac{\partial e'}{\partial L'} + \Delta G \frac{\partial e'}{\partial G'} + \frac{1}{2} (\Delta L)^2 \frac{\partial^2 e'}{\partial L'^2} + \frac{1}{2} (\Delta G)^2 \frac{\partial^2 e'}{\partial G'^2} + \Delta L \Delta G \frac{\partial^2 e'}{\partial L' \partial G'} \\
& + \frac{1}{6} (\Delta L)^3 \frac{\partial^3 e'}{\partial L'^3} + \frac{1}{6} (\Delta G)^3 \frac{\partial^3 e'}{\partial G'^3} + \frac{1}{2} (\Delta L)^2 \Delta G \frac{\partial^3 e'}{\partial L'^2 \partial G'} + \frac{1}{2} \Delta L (\Delta G)^2 \frac{\partial^3 e'}{\partial L' \partial G'^2}, \quad (7)
\end{aligned}$$

where

$$\frac{\partial \mathbf{e}'}{\partial \mathbf{L}'} = \frac{1}{\mathbf{e}' \sqrt{\mathbf{a}'}}$$

$$\frac{\partial \mathbf{e}'}{\partial \mathbf{G}'} = -\frac{1}{\mathbf{e}' \sqrt{\mathbf{a}'}}$$

$$\frac{\partial^2 \mathbf{e}'}{\partial \mathbf{L}'^2} = -\frac{1 + \mathbf{e}'^2}{\mathbf{e}'^3 \mathbf{a}'}$$

$$\frac{\partial^2 \mathbf{e}'}{\partial \mathbf{G}'^2} = -\frac{1}{\mathbf{e}'^3 \mathbf{a}'}$$

$$\frac{\partial^2 \mathbf{e}'}{\partial \mathbf{L}' \partial \mathbf{G}'} = -\frac{1 + \frac{1}{2} \mathbf{e}'^2}{\mathbf{e}'^3 \mathbf{a}'}$$

$$\frac{\partial^3 \mathbf{e}'}{\partial \mathbf{L}'^3} = \frac{3(1 + \mathbf{e}'^4)}{\mathbf{e}'^5 \mathbf{a}' \sqrt{\mathbf{a}'}}$$

$$\frac{\partial^3 \mathbf{e}'}{\partial \mathbf{G}'^3} = -\frac{3 \left(1 - \frac{1}{2} \mathbf{e}'^2 + \frac{1}{8} \mathbf{e}'^4 \right)}{\mathbf{e}'^5 \mathbf{a}' \sqrt{\mathbf{a}'}}$$

$$\frac{\partial^3 \mathbf{e}'}{\partial \mathbf{L}'^2 \partial \mathbf{G}'} = -\frac{3 - \frac{1}{2} \mathbf{e}'^2 + \frac{9}{8} \mathbf{e}'^4}{\mathbf{e}'^5 \mathbf{a}' \sqrt{\mathbf{a}'}}$$

$$\frac{\partial^3 \mathbf{e}'}{\partial \mathbf{L}' \partial \mathbf{G}'^2} = -\frac{3 - \mathbf{e}'^2}{\mathbf{e}'^5 \mathbf{a}' \sqrt{\mathbf{a}'}}$$

Expressions for $\wedge \mathbf{L}$ and $\wedge \mathbf{G}$ are now needed.

A transformation from unprimed to primed variables expands the relations

$$\begin{aligned}\mathbf{L} &= \mathbf{L}' + \frac{\partial \mathbf{S}_1}{\partial \ell} + \frac{\partial \mathbf{S}_2}{\partial \ell} \\ \mathbf{G} &= \mathbf{G}' + \frac{\partial \mathbf{S}_1}{\partial g} + \frac{\partial \mathbf{S}_2}{\partial g}\end{aligned}\quad (8)$$

to

$$\begin{aligned}\mathbf{L} &= \mathbf{L}' + \frac{\partial \mathbf{S}_1}{\partial \ell'} + \frac{\partial \mathbf{S}_2}{\partial \ell'} + \frac{\partial^2 \mathbf{S}_1}{\partial \ell'^2} \left[-\frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{L}' \partial \ell'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{L}' \partial g'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} \right] \\ &\quad + \frac{\partial^2 \mathbf{S}_1}{\partial \ell' \partial g'} \left[-\frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{G}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial \ell'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial g'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} \right] + \frac{1}{2} \frac{\partial^3 \mathbf{S}_1}{\partial \ell'^3} \left(\frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} \right)^2 \\ &\quad + \frac{1}{2} \frac{\partial^3 \mathbf{S}_1}{\partial \ell' \partial g'^2} \left(\frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} \right)^2 + \frac{\partial^3 \mathbf{S}_1}{\partial \ell'^2 \partial g'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} - \frac{\partial^2 \mathbf{S}_2}{\partial \ell'^2} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} - \frac{\partial^2 \mathbf{S}_2}{\partial \ell' \partial g'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} \\ \mathbf{G} &= \mathbf{G}' + \frac{\partial \mathbf{S}_1}{\partial g'} + \frac{\partial \mathbf{S}_2}{\partial g'} + \frac{\partial^2 \mathbf{S}_1}{\partial g'^2} \left[-\frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{G}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial \ell'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{G}' \partial g'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} \right] \\ &\quad + \frac{\partial^2 \mathbf{S}_1}{\partial g' \partial \ell'} \left[-\frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} - \frac{\partial \mathbf{S}_2}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{L}' \partial \ell'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} + \frac{\partial^2 \mathbf{S}_1}{\partial \mathbf{L}' \partial g'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} \right] + \frac{1}{2} \frac{\partial^3 \mathbf{S}_1}{\partial g'^3} \left(\frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} \right)^2 \\ &\quad + \frac{1}{2} \frac{\partial^3 \mathbf{S}_1}{\partial g' \partial \ell'^2} \left(\frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} \right)^2 + \frac{\partial^3 \mathbf{S}_1}{\partial g'^2 \partial \ell'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} - \frac{\partial^2 \mathbf{S}_2}{\partial g'^2} \frac{\partial \mathbf{S}_1}{\partial \mathbf{G}'} - \frac{\partial^2 \mathbf{S}_2}{\partial g' \partial \ell'} \frac{\partial \mathbf{S}_1}{\partial \mathbf{L}'} ,\end{aligned}\quad (9)$$

where, again, f and g are to be replaced by f' and g' .

Making use of relations (4), Equations (9) become

$$\mathbf{L} = \mathbf{L}' + \Delta \mathbf{L},$$

where

$$\begin{aligned}
 \Delta L = & \frac{J_2}{32a' \sqrt{a'}} \left\{ -2(1-3\cos^2 i') [12e'^2 + 3e'(4+13e'^2)\cos f' + 6e'^2 \cos 2f' + e'^3 \cos 3f'] \right. \\
 & + 3\sin^2 i' [6e'^2 \cos 2g' + e'(12+43e'^2)\cos(f'+2g') + e'^3 \cos(f'-2g')] \\
 & + 4(2+9e'^2)\cos 2(f'+g') + e'(12+43e'^2)\cos(3f'+2g') + 6e'^2 \cos(4f'+2g') \\
 & \left. + e'^3 \cos(5f'+2g') \right\} \\
 & + \frac{3J_2^2}{64a'^3 \sqrt{a'}} \left\{ 12\sin^2 i' (1-5\cos^2 i')(f'-l') [3e'\sin(f'+2g') + 2\sin 2(f'+g') + 3e'\sin(3f'+2g')] \right. \\
 & + 11-34\cos^2 i' + 47\cos^4 i' + 2e'(29-90\cos^2 i' + (45\cos^4 i')\cos f' \\
 & - e'\sin^2 i' (61-99\cos^2 i')\cos(f'+2g') - 8\sin^2 i' (4-9\cos^2 i')\cos 2(f'+g') \\
 & - 5e'\sin^2 i' (17-47\cos^2 i')\cos(3f'+2g') + 26e'\sin^4 i' \cos(3f'+4g') \\
 & \left. + 5\sin^4 i' \cos 4(f'+g') + 18e'\sin^4 i' \cos(5f'+4g') \right\} \\
 & + \frac{3J_2^3}{4096e'a'^5 \sqrt{a'}} \left\{ -4(1-3\cos^2 i')(97-266\cos^2 i' + 241\cos^4 i')\cos f' \right. \\
 & + 36(1-3\cos^2 i')(9-26\cos^2 i' + 25\cos^4 i')\cos 3f' - \sin^2 i' (263-1150\cos^2 i' + 1511\cos^4 i')\cos(f'+2g') \\
 & - 3\sin^2 i' (19-86\cos^2 i' + 115\cos^4 i')\cos(f'-2g') - 18\sin^4 i' (1-3\cos^2 i')\cos(f'+4g') \\
 & + \sin^2 i' (1081-4610\cos^2 i' + 5977\cos^4 i')\cos(3f'+2g') + 306\sin^4 i' (1-3\cos^2 i')\cos(3f'+4g') \\
 & + 9\sin^6 i' \cos(3f'+6g') - 35\sin^2 i' (19-86\cos^2 i' + 115\cos^4 i')\cos(5f'+2g') \\
 & - 910\sin^4 i' (1-3\cos^2 i')\cos(5f'+4g') - 105\sin^6 i' \cos(5f'+6g') \\
 & \left. + 686\sin^4 i' (1-3\cos^2 i')\cos(7f'+4g') + 343\sin^6 i' \cos(7f'+6g') - 343\sin^6 i' \cos(9f'+6g') \right\}
 \end{aligned}$$

$$G = G' \perp \wedge G,$$

where

$$\begin{aligned}
 \Delta G = & \frac{J_2 \sin^2 i'}{8a' \sqrt{a'}} (2 + 3e'^2) [3e' \cos(f' + 2g') + 3 \cos 2(f' + g') + e' \cos(3f' + 2g')] \\
 & + \frac{J_2^2 \sin^2 i'}{32a' \sqrt{a'}} \{ 12(1 - 5\cos^2 i')(f' - l') [3e' \sin(f' + 2g') + 3 \sin 2(f' + g') + e' \sin(3f' + 2g')] \\
 & - (7 - 25\cos^2 i') - 24e'(1 - 3\cos^2 i') \cos f' + 9e'(1 - 19\cos^2 i') \cos(f' + 2g') \\
 & - 18 \sin^2 i' \cos 2(f' + g') - e'(29 - 103\cos^2 i') \cos(3f' + 2g') + 3e' \sin^2 i' \cos(3f' + 4g') \\
 & - 3 \sin^2 i' \cos 4(f' + g') - 3e' \sin^2 i' \cos(5f' + 4g') \} \\
 & + \frac{J_2^3 \sin^2 i'}{2048e'a' \sqrt{a'}} \{ -3(263 - 1150\cos^2 i' + 1511\cos^4 i') \cos(f' + 2g') \\
 & + 9(19 - 86\cos^2 i' + 115\cos^4 i') \cos(f' - 2g') - 108 \sin^2 i' (1 - 3\cos^2 i') \cos(f' + 4g') \\
 & + (1081 - 4610\cos^2 i' + 5977\cos^4 i') \cos(3f' + 2g') + 612 \sin^2 i' (1 - 3\cos^2 i') \cos(3f' + 4g') \\
 & + 27 \sin^4 i' \cos(3f' + 6g') - 21(19 - 86\cos^2 i' + 115\cos^4 i') \cos(5f' + 2g') \\
 & - 1092 \sin^2 i' (1 - 3\cos^2 i') \cos(5f' + 4g') - 189 \sin^4 i' \cos(5f' + 6g') \\
 & + 588 \sin^2 i' (1 - 3\cos^2 i') \cos(7f' + 4g') + 441 \sin^4 i' \cos(7f' + 6g') - 343 \sin^4 i' \cos(9f' + 6g') \} \quad (10)
 \end{aligned}$$

Equation (7) then becomes

$$\begin{aligned}
 e = e' + \frac{J_2}{8a'^2} & [-6(1-3\cos^2 i')\cos f' + 3\sin^2 i'\cos(f'+2g') + 7\sin^2 i'\cos(3f'+2g')] \\
 + \frac{J_2^2}{256e'^2a'^4} & [2(47-166\cos^2 i'+191\cos^4 i') + 36\sin^2 i'(1-3\cos^2 i')\cos 2g' \\
 - 6(13-50\cos^2 i'+61\cos^4 i')\cos 2f' - 120\sin^2 i'(1-3\cos^2 i')\cos 2(f'+g') \\
 - 9\sin^4 i'\cos(2f'+4g') + 84\sin^2 i'(1-3\cos^2 i')\cos(4f'+2g') \\
 + 42\sin^4 i'\cos 4(f'+g') - 49\sin^4 i'\cos(6f'+4g')] ; \quad (11)
 \end{aligned}$$

all terms of the type $\frac{J_2^3}{e'^2}$ have dropped out.

LONG PERIOD PERTURBATIONS

The long period perturbations in g and e , as presented in Reference 3, are

$$\begin{aligned}
 e' - e_1 &= \left(Q - \frac{Q^3}{8e_1^2} \right) \sin g'' + \frac{Q^2}{4e_1} \cos 2g'' - \frac{Q^3}{8e_1^2} \sin 3g'' \\
 g' - g'' &= \left(\frac{Q}{e_1} + \frac{Q^3}{4e_1^3} \right) \cos g'' + \left[\frac{Q^2}{2e_1^2} + \frac{9J_2^2 \sin^2 i'' (1-3\cos^2 i'')}{32e_1^2 a'^4} \right] \sin 2g'' \\
 &\quad - \left[\frac{Q^3}{3e_1^3} + \frac{9J_2^2 \sin^2 i'' (1-3\cos^2 i'')}{16e_1^3 a'^4} Q \right] \cos 3g'' . \quad (12)
 \end{aligned}$$

where

$$e_1 \approx e'' \left(1 + \frac{Q^2}{4e''^2} \right)$$

$$Q = \frac{M}{N}$$

$$N = -\frac{3J_2}{4a''^3\sqrt{a''}}(1 - 5\cos^2 i'') + \frac{3J_2^2}{64a''^5\sqrt{a''}}(7 - 114\cos^2 i'' + 395\cos^4 i'')$$

$$- \frac{15J_4}{32a''^5\sqrt{a''}}(3 - 36\cos^2 i'' + 49\cos^4 i'') + \dots$$

$$M = \frac{3J_3 \sin i''}{8a''^4\sqrt{a''}}(1 - 5\cos^2 i'') + \frac{15J_5 \sin i''}{32a''^6\sqrt{a''}}(1 - 14\cos^2 i'' + 21\cos^4 i'')$$

$$+ \frac{105J_7 \sin i''}{1024a''^8\sqrt{a''}}(5 - 135\cos^2 i'' + 495\cos^4 i'' - 429\cos^6 i'') + \dots$$

The expression for ℓ will be the negative of that for g , so that no small divisors of e occur in $\ell + g$.

However, Equation (11) indicates that an additional constant and long period term appears in the expression for e . Designating

$$\bar{\Delta}e = \frac{J_2^2}{128e'a'^4}[47 - 166\cos^2 i' + 191\cos^4 i' + 18\sin^2 i'(1 - 3\cos^2 i')\cos 2g'] \quad (13)$$

it is desirable to transform $\bar{\Delta}e$ to double primed variables by

$$\bar{\Delta}e = \bar{\Delta}e(L'', G'', H'', g'') + \frac{\partial(\bar{\Delta}e)}{\partial G''} \frac{\partial(\Delta S_1^*)}{\partial g'} - \frac{\partial(\bar{\Delta}e)}{\partial g''} \frac{\partial(\Delta S_1^*)}{\partial G''}, \quad (14)$$

where

$$\Delta S_1^* = e''\sqrt{a''} Q \cos g'$$

$$\frac{\partial(\Delta S_1^*)}{\partial g'} = -e''\sqrt{a''} Q \sin g''$$

$$\frac{\partial(\Delta S_1^*)}{\partial G''} = -\frac{Q}{e''} \cos g'.$$

Since

$$\frac{\partial(\bar{\Delta}e)}{\partial G''} = \frac{J_2^2}{128e''^3 a''^4 \sqrt{a''}} [47 - 166 \cos^2 i'' + 191 \cos^4 i'' + 18 \sin^2 i'' (1 - 3 \cos^2 i'') \cos 2g'']$$

$$-\frac{\partial(\bar{\Delta}e)}{\partial g''} = \frac{9 J_2^2 \sin^2 i'' (1 - 3 \cos^2 i'')}{32 e'' a''^4} \sin 2g'',$$

Equation (14) becomes

$$\bar{\Delta}e = \frac{J_2^2}{128e'' a''^4} [47 - 166 \cos^2 i'' + 191 \cos^4 i'' + 18 \sin^2 i'' (1 - 3 \cos^2 i'') \cos 2g'']$$

$$-\frac{J_2^2}{64e''^2 a''^4} (28 - 101 \cos^2 i'' + 218 \cos^4 i'') Q \sin g'' - \frac{27 J_2^2 \sin^2 i'' (1 - 3 \cos^2 i'')}{128e''^2 a''^4} Q \sin 3g''. \quad (15)$$

Thus, a more complete long period expression for e' is the following:

$$e' = e_1 + \frac{J_2^2}{128e_1 a''^4} (47 - 166 \cos^2 i'' + 191 \cos^4 i'')$$

$$+ \left[Q - \frac{Q^3}{8e_1^2} - \frac{J_2^2}{64e_1^2 a''^4} (28 - 101 \cos^2 i'' + 218 \cos^4 i'') Q \right] \sin g''$$

$$+ \left[\frac{Q^2}{4e_1} + \frac{9 J_2^2}{64e_1 a''^4} \sin^2 i'' (1 - 3 \cos^2 i'') \right] \cos 2g''$$

$$- \left[\frac{Q^3}{8e_1^2} + \frac{27 J_2^2}{128e_1^2 a''^4} \sin^2 i'' (1 - 3 \cos^2 i'') Q \right] \sin 3g''. \quad (16)$$

In Reference 3, values of eccentricity and argument of perigee (after correcting for lunar and solar effects) for Alouette 1 and Tiros 8 were fit, by means of least squares, to trigonometric series with g'' as the basic argument. The results were the following:

Alouette 1

$$e' = .0025164 + .0011119 \sin g'' + .00013369 \cos 2g'' - .0000097969 \sin 3g''$$

$$g' = 0.30972 + 0.45602 \cos g'' - 0.080879 \sin 2g'' - 0.0032704 \cos 3g'' \text{ (radians)} \quad (17)$$

Tiros 8

$$e' = .0034395 + .0015519 \sin g'' + .00013896 \cos 2g'' - .000016406 \sin 3g''$$

$$g' = 2.19311 + 0.46092 \cos g'' - 0.061533 \sin 2g'' - 0.0093447 \cos 3g'' \text{ (radians)} \quad (18)$$

Comparison of the $\sin g''$ and $\cos g''$ terms in Equation (17) and (18) with those in Equations (12) and (16) yields the following:

Alouette 1

$$\frac{Q}{e_1} + \frac{Q^3}{4e_1^3} = 0.45602 \Rightarrow Q = 0.43539 e_1$$

$$Q - \frac{Q^3}{8e_1^2} - \frac{J_2^2}{64e_1^2 a''^4} (28 - 101 \cos^2 i'' + 218 \cos^4 i'') Q = 0.0011119, Q = 0.43539 e_1 \Rightarrow e_1 = 0.0027131, Q = 0.001181$$

Tiros 8

$$\frac{Q}{e_1} + \frac{Q^3}{4e_1^3} = 0.46092 \Rightarrow Q = 0.43967 e_1$$

$$Q - \frac{Q^3}{8e_1^2} - \frac{J_2^2}{64e_1^2 a''^4} (28 - 101 \cos^2 i'' + 218 \cos^4 i'') Q = 0.0015519, Q = 0.43967 e_1 \Rightarrow e_1 = 0.0036724, Q = 0.0016146$$

In Reference 3, Kozai's (Reference 4) values for the zonal harmonic coefficients through J_{11} were used to compute the following "theoretical" values for Q:

Alouette 1: $Q = 0.0011183$

Tiros 8: $Q = 0.0015869$

These compare favorably with the experimental values.

The values of e_1 and Q can then be substituted into Equations (12) and (16) to obtain predicted amplitudes for the $2g''$ and $3g''$ terms:

Alouette 1

$$\cos 2g'': \frac{Q^2}{4e_1} + \frac{9J_2^2 \sin^2 i'' (1 - 3\cos^2 i'')}{64e_1^4 a''^4} = 0.00015864 \text{ (observed: 0.00013369)}$$

$$\sin 3g'': -\frac{Q^3}{8e_1^2} - \frac{27J_2^2 \sin^2 i'' (1 - 3\cos^2 i'')}{128e_1^2 a''^4} Q = -0.000047623 \text{ (observed: -0.0000097969)}$$

$$\sin 2g'': -\frac{Q^2}{2e_1^2} - \frac{9J_2^2 \sin^2 i'' (1 - 3\cos^2 i'')}{32e_1^2 a''^4} = -0.116941 \text{ (observed: -0.080879)}$$

$$\cos 3g'': -\frac{Q^3}{3e_1^3} - \frac{9J_2^2 \sin^2 i'' (1 - 3\cos^2 i'')}{16e_1^3 a''^4} Q = -0.046807 \text{ (observed: -0.0032704)}$$

Tiros 8

$$\cos 2g'': \frac{Q^2}{4e_1} + \frac{9J_2^2 \sin^2 i'' (1 - 3\cos^2 i'')}{64e_1^4 a''^4} = 0.00018130 \text{ (observed: 0.00013896)}$$

$$\sin 3g'': -\frac{Q^3}{8e_1^2} - \frac{27J_2^2 \sin^2 i'' (1 - 3\cos^2 i'')}{128e_1^2 a''^4} Q = -0.000040608 \text{ (observed: -0.000016406)}$$

$$\sin 2g'': -\frac{Q^2}{2e_1^2} - \frac{9J_2^2 \sin^2 i'' (1 - 3\cos^2 i'')}{32e_1^2 a''^4} = -0.098743 \text{ (observed: -0.061533)}$$

$$\cos 3g'': -\frac{Q^3}{3e_1^3} - \frac{9J_2^2 \sin^2 i'' (1 - 3\cos^2 i'')}{16e_1^3 a''^4} Q = -0.030166 \text{ (observed: -0.0093447)}$$

AMPLITUDES OF SHORT PERIOD TERMS

Using mean values for the orbital elements of the small eccentricity satellites Alouette 1 and Tiros 8, and using the value 1.0826×10^{-3} for J_2 , the short period terms in Equations (5) and (11) have the following amplitudes:

Alouette 1

$$\Delta\ell(J_2 \text{ part}) = 0.22049 \sin f' + 0.11683 \sin(f' + 2g') - 0.27261 \sin(3f' + 2g') \text{ (radians)}$$

$$\begin{aligned} \Delta\ell(J_2^2 \text{ part}) = & 0.056150 \sin 2f' - 0.0068244 \sin(2f' + 4g') - 0.060101 \sin(4f' + 2g') \\ & + 0.037155 \sin(6f' + 4g') \text{ (radians)} \end{aligned}$$

$$\begin{aligned} \Delta\ell(J_2^3 \text{ part}) = & -0.0079433 \sin f' + 0.019818 \sin 3f' - 0.0054459 \sin(f' + 2g') \\ & - 0.010574 \sin(f' - 2g') - 0.0033854 \sin(f' + 4g') + 0.0074849 \sin(3f' + 2g') \\ & + 0.0021315 \sin(3f' + 4g') + 0.00059797 \sin(3f' + 6g') - 0.024674 \sin(5f' + 2g') \\ & - 0.0038034 \sin(5f' + 4g') - 0.00046509 \sin(5f' + 6g') + 0.018432 \sin(7f' + 4g') \\ & + 0.0010852 \sin(7f' + 6g') - 0.0075964 \sin(9f' + 6g') \text{ (radians)} \end{aligned}$$

$$10^4 \cdot \Delta e(J_2 \text{ part}) = -5.5481 \cos f' + 2.9399 \cos(f' + 2g') + 6.8597 \cos(3f' + 2g')$$

$$\begin{aligned} 10^4 \cdot \Delta e(J_2^2 \text{ part}) = & -0.70653 \cos 2f' - 1.0803 \cos 2(f' + g') - 0.085871 \cos(2f' + 4g') \\ & + 0.75625 \cos(4f' + 2g') + 0.40073 \cos 4(f' + g') - 0.46752 \cos(6f' + 4g') \end{aligned}$$

Tiros 8

$$\Delta\ell(J_2 \text{ part}) = 0.034424 \sin f' + 0.069146 \sin(f' + 2g') - 0.16134 \sin(3f' + 2g') \text{ (radians)}$$

$$\begin{aligned} \Delta\ell(J_2^2 \text{ part}) = & 0.011748 \sin 2f' - 0.0023905 \sin(2f' + 4g') - 0.0055539 \sin(4f' + 2g') \\ & + 0.013015 \sin(6f' + 4g') \text{ (radians)} \end{aligned}$$

$$\begin{aligned}
\Delta \ell (J_2^3 \text{ part}) = & -0.00036625 \sin f' + 0.00087933 \sin 3f' - 0.00053558 \sin (f' + 2g') \\
& - 0.00095983 \sin (f' - 2g') - 0.00018515 \sin (f' + 4g') + 0.00077573 \sin (3f' + 2g') \\
& + 0.00011658 \sin (3f' + 4g') + 0.00012397 \sin (3f' + 6g') - 0.0022396 \sin (5f' + 2g') \\
& - 0.00020801 \sin (5f' + 4g') - 0.000096417 \sin (5f' + 6g') + 0.0010080 \sin (7f' + 4g') \\
& + 0.00022497 \sin (7f' + 6g') - 0.0015748 \sin (9f' + 6g') \text{ (radians)}
\end{aligned}$$

$$10^4 \times \Delta e (J_2 \text{ part}) = -1.1841 \cos f' + 2.3784 \cos (f' + 2g') + 5.5494 \cos (3f' + 2g')$$

$$\begin{aligned}
10^4 \times \Delta e (J_2^2 \text{ part}) = & -0.20202 \cos 2f' - 0.13644 \cos 2(f' + g') - 0.041108 \cos (2f' + 4g') \\
& + 0.095507 \cos (4f' + 2g') + 0.19184 \cos 4(f' + g') - 0.22381 \cos (6f' + 4g')
\end{aligned}$$

Clearly, the J_2^2 and J_2^3 terms are comparable to the J_2 terms; certainly, there is not a 10^{-3} factor difference between terms of succeeding "orders" of J_2 .

IN-TRACK, CROSS-TRACK, ALONG-TRACK ERRORS

Consider a coordinate system defined in the following manner: x and y axes in the directions of the position and velocity vectors of the satellite, respectively (and therefore in the orbital plane of the satellite), and the z axis perpendicular to the orbit plane. The in-track, along-track, and cross-track errors in the position of the satellite are simply those errors in the x, y, and z directions, respectively, and are represented by the following:

In-track: Δr

Along-track: $r(\Delta h \cos i + \Delta f + \Delta g)$

Cross-track: $r[\sin(f + g)\Delta i + \sin i \cos(f + g)\Delta h]$

Since no terms with eccentricity as a divisor appear in Δi and Δh , and since all such terms cancel out in the sum $\Delta f + \Delta g$, it is apparent that these small divisors cause no along-track or cross-track errors. However, it is not so obvious that no in-track errors result, also.

The radial distance r of the satellite from the center of the earth can be expressed as a function of ℓ , L , and G . In order to express r as a function of primed variables, the following expansion is required:

$$\begin{aligned} \Delta r = & \Delta \ell \frac{\partial r}{\partial \ell'} + \Delta L \frac{\partial r}{\partial L'} + \Delta G \frac{\partial r}{\partial G'} + \frac{1}{2} (\Delta \ell)^2 \frac{\partial^2 r}{\partial \ell'^2} + \frac{1}{2} (\Delta L)^2 \frac{\partial^2 r}{\partial L'^2} + \frac{1}{2} (\Delta G)^2 \frac{\partial^2 r}{\partial G'^2} \\ & + \Delta \ell \Delta L \frac{\partial^2 r}{\partial \ell' \partial L'} + \Delta \ell \Delta G \frac{\partial^2 r}{\partial \ell' \partial G'} + \Delta L \Delta G \frac{\partial^2 r}{\partial L' \partial G'} \end{aligned} \quad (19)$$

where

$$\frac{\partial r}{\partial \ell'} = a' e' \sin f'$$

$$\frac{\partial r}{\partial L'} = -\frac{\sqrt{a'}}{e'} \cos f' + 2\sqrt{a'}$$

$$\frac{\partial r}{\partial G'} = \frac{\sqrt{a'}}{e'} \cos f'$$

$$\frac{\partial^2 r}{\partial \ell'^2} = a' e' \cos f'$$

$$\frac{\partial^2 r}{\partial L'^2} = \frac{1}{e'^2} + \left(\frac{1}{e'^3} - \frac{11}{4e'} \right) \cos f' - \frac{1}{e'^2} \cos 2f' - \frac{1}{4e'} \cos 3f'$$

$$\frac{\partial^2 r}{\partial G'^2} = \frac{1}{e'^2} + \left(\frac{1}{e'^3} + \frac{1}{4e'} \right) \cos f' - \frac{1}{e'^2} \cos 2f' - \frac{1}{4e'} \cos 3f'$$

$$\frac{\partial^2 r}{\partial \ell' \partial L'} = \frac{\sqrt{a'}}{e'} \sin f' + \sqrt{a'} \sin 2f'$$

$$\frac{\partial^2 r}{\partial \ell' \partial G'} = -\frac{\sqrt{a'}}{e'} \sin f' + \sqrt{a'} \sin 2f'$$

$$\frac{\partial^2 r}{\partial L' \partial G'} = -\frac{1}{e'^2} + \left(\frac{1}{e'^3} - \frac{5}{4e'} \right) \cos f' + \frac{1}{e'^2} \cos 2f' + \frac{1}{4e'} \cos 3f' \quad (20)$$

and $\Delta\ell$, ΔL , and ΔG appear in Equations (5) and (10). All terms with e' as a divisor cancel; the only remaining terms of order J_2 are the following:

$$\Delta r = \frac{J_2}{4a'} [3(1 - 3\cos^2 i') + \sin^2 i' \cos 2(f' + g')] . \quad (21)$$

Thus, even though some of the orbital elements of a small eccentricity satellite appear to be affected by small eccentricity divisors appearing in perturbation expressions, no such "singularities" appear in the coordinates.

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Appendix

SYMBOLS

a = semi-major axis of satellite's orbit

e = eccentricity of satellite's orbit

i = inclination of satellite's orbital plane to earth's equatorial plane

α = mean anomaly of satellite

g = argument of perigee of satellite

h = longitude of ascending node of satellite's orbit

f = true anomaly of satellite

r = radial distance from the center of the earth to the satellite

μ = product of gravitational constant with the mass of the earth

$$L = \sqrt{\mu/a}$$

$$G = L \cdot \sqrt{1 - e^2}$$

$$H = G \cos i$$

L' , L'' , f' , f'' , etc. = variables used in von Zeipel method

$$a' = \mu^{-1} L'^2$$

$$a'' = \mu^{-1} L''^2$$

$$e' = \sqrt{1 - \frac{G'^2}{L'^2}}$$

$$e'' = \sqrt{1 - \frac{G''^2}{L''^2}}$$

$$i' = \cos^{-1} \left(\frac{H'}{G'} \right)$$

$$i'' = \cos^{-1} \left(\frac{H''}{G''} \right)$$

e_1 = eccentricity constant

J_2, J_3 , etc. = zonal harmonic coefficients in earth's gravitational potential

S_1, S_2 = short period determining functions

ΔS_1^* = long period determining function